NOTATION

R, R_p, R_0, radii of the deformation, ingot, and heating element zones; τ, time; μ, μ_0, melt viscosity and the viscosity scale when going over to dimensionless variables; x, longitudinal coordinate; T, T_0, T_p, temperatures of the melt, the gas being blown through the heating zone, and the furnace; T_{p1}, T_{p2}, maximal and minimal heating element temperature; λ_r, λ_e, melt and effective coefficient of molecular conductivity which takes account of both the molecular and the radiant conduction; β, reflection coefficient; ε_p, ε, emissivities of the heating element and the melt; η, an integration variable; l, heating element length; ρ, c, melt density and specific heat; h, coefficient of external heat elimination; n_c, refractive index of the gas being blown through the heating zone; σ_0, Stefan–Boltzmann constant; σ, coefficient of surface tension; d_1, d_2, coefficients in the temperature dependence of the viscosity; $\bar{a}_2$, a_1, geometric dimensions of the heating zone; N, quantity of points along the longitudinal coordinate during discretization of the design domain; V, V_H, V_0, velocities of the melt, the ingot delivery, and the drawing.

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TEMPERATURE-MEASUREMENT OPTIMIZATION AND NUMERICAL INVERSE CONDUCTION-TREATMENT SOLUTION

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Practical evidence is given that locally optimal measurement planning can be applied in nonstationary thermophysical experiments.

Inverse treatments in thermophysics require a preliminary examination of topics in the formalization and algorithmization, as well as choice of working conditions to provide high accuracy. Simulation results [1] show that the systematic error in solving an inverse treatment is substantially dependent on the number of sensors used in the measurements and the positions of them even if the exact values are known for the measured temperatures. A measurement scheme exists for which one can determine the unknown behavior of the thermophysical characteristics accurately. Measurement plan optimization before the experiments is therefore of interest. One can use experiment planning theory [2].

Optimum temperature-measurement planning is based on the following. We introduce a measurement plan

$$\xi = \{N, \bar{X}\}, \bar{X} = \{X_i\}_1^N.$$

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We construct a scalar performance criterion $\psi(\xi)$ dependent on the plan ξ and characterizing the accuracy in solving the inverse treatment, and then we consider the definition of the optimal plan

$$\xi_0 = \operatorname{Arg}\max_{\xi \in \Xi} \psi(\xi), \tag{1}$$

where Ξ is the permissible plan set.

In (1), one cannot construct a criterion that directly defines the accuracy, and therefore one uses indirect quality criteria; constructive results can be obtained when the unknown functions are represented parametrically and the inverse treatment amounts to determining the unknown parameters $\bar{P} = \{p_k\}_1^m$. Then the most common indirect criterion is the determinant for the normalized Fisher information matrix [2]:

$$F(\xi) = \frac{1}{N}\{\Phi_{k,j}\}, \quad k, j = \overline{1, m}, \tag{2}$$

where

$$\Phi_{k,j} = \sum_{i=1}^{N} \int_0^{\tau_m} \mu_i(\tau)\,\theta_k(X_i, \tau)\,\theta_j(X_i, \tau)\,d\tau,$$

and $\theta_k(x, \tau)$, $k = \overline{1, m}$ are sensitivity functions, while $\mu_i(\tau)$, $i = \overline{1, N}$ are measurement performance functions, which enable one to incorporate the errors in the measurements; $\mu_i(\tau) = \sigma^{-2}$ for equally accurate measurements, where σ^{-2} is based on the measurement-error variance.

There is an essentially nonlinear relation between the temperature and the unknown parameters in an inverse thermal-conduction treatment; therefore, the $\theta_k(x, \tau)$, $k = \overline{1, m}$ and the elements in (2) together with ξ are dependent on the unknown parameter vector $\bar{P}$, and only locally optimal planning is possible, where one solves (1) by means of a priori information on the unknown parameters [2].

Optimum planning can be used in thermophysics if two basic concepts can be demonstrated: 1) the performance criterion is independent for the problem class, and 2) locally optimal planning is possible subject to indefinite information on the unknown parameters. The main purpose of this paper is to give practical evidence for these concepts from computational experiments.

The study relates to the temperature dependence of the conductivity for an insulating material. We consider one-dimensional nonstationary conduction subject to boundary conditions of the first kind. One has to determine $\lambda(T)$ from

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right), \quad 0 < x < b, \quad 0 < \tau \leqslant \tau_m, \tag{3}$$

$$T(x, 0) = T_0(x), \quad 0 \leqslant x \leqslant b, \tag{4}$$

$$T(0, \tau) = g_1(\tau), \tag{5}$$

$$T(b, \tau) = g_2(\tau), \tag{6}$$

$$T^{\exp}(X_i, \tau) = f_i(\tau), \quad i = \overline{1, N}, \quad 0 < X_1 < X_2 < \ldots < X_N < b, \tag{7}$$

where $C(T)$, $T_0(x)$, $g_1(\tau)$, $g_2(\tau)$ are given functions.

We first consider temperature-measurement planning, where the input data are: specimen thickness b taken as b = 0.03 m, initial temperature distribution constant, $T_0(x) = 300$ K, and experiment duration $\tau_m = 700$ sec. The thermophysical characteristics in (3) were taken as close to the [3] ones, being taken as known and specified by:

$$C(T) = 1.0 \cdot 10^5 + 2.90T - 0.31T^2 + 0.11 \cdot 10^{-3}T^3, \quad \lambda(T) = 0.39 \cdot 10^{-1} + 0.17 \cdot 10^{-3}T. \tag{8}$$

The boundary conditions are represented in Fig. 1 ($\bar{T} = T/T_{max}$, $\bar{\tau} = \tau/\tau_m$).

The unknown function is parameterized as

$$\lambda(T) = \sum_{k=1}^{m} \lambda_k \varphi_k(T), \tag{9}$$

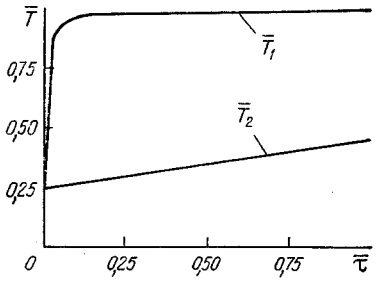

Fig. 1. Temperature variation at specimen boundaries.

where $\varphi_k(T)$, $k=\overline{1, m}$ is a basis-function system, for which we used cubic B splines [4]. To represent (9), we specified natural boundary conditions: $\lambda''(T_{min}) = \lambda''(T_{max}) = 0$. The minimum temperature T_{min} and the maximum T_{max} were determined from the given boundary conditions. The degree of the spline was chosen to provide continuity in the coefficients in the sensitivity-function equation. The formulas defining the sensitivity functions were derived by differentiating (4)-(7) m times with respect to λ_k, $k = \overline{1, m}$, and are

$$C(T)\frac{\partial\theta_k}{\partial\tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial\theta_k}{\partial x}\right) + \frac{\partial\lambda}{\partial T}\frac{\partial T}{\partial x}\frac{\partial\theta_k}{\partial x} + \left[\frac{\partial\lambda}{\partial T}\frac{\partial^2 T}{\partial x^2} + \right.$$
$$\left. + \frac{\partial^2\lambda}{\partial T^2}\left(\frac{\partial T}{\partial x}\right)^2 - \frac{\partial C}{\partial T}\frac{\partial T}{\partial\tau}\right]\theta_k + \frac{\partial^2 T}{\partial x^2}\varphi_k(T) + \left(\frac{\partial T}{\partial x}\right)\frac{d\varphi_k}{dT}, \quad (10)$$

$$0 < x < b,\ 0 < \tau \leqslant \tau_m,\ k = \overline{1, m},$$

$$\theta_k(x, 0) = 0,\ 0 \leqslant x \leqslant b, \quad (11)$$

$$\theta_k(0, \tau) = 0, \quad (12)$$

$$\theta_k(b, \tau) = 0. \quad (13)$$

We solved (3)-(6) and (10)-(13) numerically by means of a monotone inexplicit approximation scheme [5].

To solve (1), the permissible-planned set Ξ was formulated from a study on the unambiguous solubility of this inverse treatment in the form of uniqueness theorems for a given type of boundary condition (see for example [6]), where it was represented as

$$\Xi = \{(N, X): N \geqslant N_{min},\ 0 < X_i < b,\ i = \overline{1, N}\}.$$

The solution to (2) was constructed sequentially; beginning with $N_{min} = 1$, we increased the number of sensors by one, and for each given N we determined the optimum measurement-point coordinate vector $\bar{X}$ by a scanning method based on a spatial finite-difference set [7, 8].

Then the [9] method was applied to the inverse problem, which was considered in a turning-point formulation; the target functional was the mean-square discrepancy:

$$I = \sum_{i=1}^{N}\int_0^{\tau_m}[T(X_i, \tau) - f_i(\tau)]^2\, d\tau. \quad (14)$$

We minimized (14) by the conjugate-gradient method; the approximations for the unknown parameters were:

$$\lambda_k^{(l+1)} = \lambda_k^{(l)} + \alpha^{(l)} g_k^{(l)},\ k = \overline{1, m},\ l = 0, 1, 2, \ldots,\ g_k^{(l)} = -I_k'^{(l)} + \beta^{(l)} g_k^{(l-1)},$$

$$\beta_0 = 0,\ \beta^{(l)} = \sum_{k=1}^{m}(I_k'^{(l)} - I_k'^{(l-1)})\, I_k'^{(l)} \Big/ \sum_{k=1}^{m}(I'^{(l-1)})^2.$$

The expression for the components of the gradient vector for the target functional was derived by means of conjugate boundary-value treatment [10]:

$$I_k' = \sum_{i=1}^{N+1}\int_{X_{i-1}}^{X_i}\int_0^{\tau_m}\nu_i(x, \tau)\left[\frac{\partial^2 T}{\partial x^2}\varphi_k(T) + \left(\frac{\partial T}{\partial x}\right)^2\frac{d\varphi_k(T)}{dT}\right]d\tau dx,\ k = \overline{1, m},$$

TABLE 1. Maximal Values for the Planning Criterion and Optimal Sensor Coordinates

ψ_{max}	x_1^*, m	x_2^*, m	x_3^*, m
$0{,}5559 \cdot 10^{37}$	0,01275	—	—
$0{,}1362 \cdot 10^{40}$	0,00750	0,02400	—
$0.1800 \cdot 10^{40}$	0.00600	0.02250	0,02400

where X_i, $i = \overline{1, N}$ are the sensor coordinates, $X_0 = 0$, $X_{N+1} = b$; $\nu_i(x, \tau)$, $i = \overline{1, N+1}$ being the conjugate variable.

A linear estimate was used for the descent depth $\alpha^{(l)}$ in iteration l from the boundary-value solution for the temperature variation $\vartheta(x, \tau)$; the formula for $\alpha^{(l)}$ was

$$\alpha^{(l)} = \frac{\sum_{i=1}^{N} \int_0^{\tau_m} [T(X_i, \tau, \bar{\lambda}^{(l)}) - f_i(\tau)] \vartheta(X_i, \tau) d\tau}{\sum_{i=1}^{N} \int_0^{\tau_m} [\vartheta(X_i, \tau)]^2 d\tau}.$$

The formulas defining the boundary-value treatments for $\nu_i(x, \tau)$, $i = \overline{1, N+1}$ and $\vartheta(x, \tau)$ have been given in [9].

We examined the rational measurement scheme for (3)-(7) on the assumption that the measurements on the nonstationary temperatures $f_i(\tau)$, $i = 1, N$, are made without error; in that idealized situation, the iteration may be halted from the condition

$$\text{abs}\,[(\bar{\lambda}^{(l)} - \bar{\lambda}^{(l-1)})/\bar{\lambda}^{(l)}] < \varepsilon,$$

where $\varepsilon > 0$ is a small quantity.

The boundary-value treatments for $T(x, \tau)$, $\nu(x, \tau)$ and $\vartheta(x, \tau)$ were handled numerically; the effective experimental values for the temperatures at various distances from the heated surface were derived by solving the (3)-(6) heating problem. The calculated temperatures $f_i(\tau)$, $i = \overline{1, N}$ at points with given coordinates X_i, $i = \overline{1, N}$ were then used as input data for the inverse derivation of the unknown $\lambda(T)$; the initial approximation was $\lambda_k^0 = 0.02$ W/(m•K), $k = 1, m$.

This procedure enables us to compare directly the known behavior of the conductivity with those recovered from the inverse treatment with various plans; the results were examined from the error in relation to the measurement plan and used to evaluate the viability and performance in choosing locally optimal plans.

Table 1 gives the results on the optimum plan; Fig. 2 illustrates the sensitivity analysis. The a priori information on $\lambda(T)$ was provided by (8).

For $N = 1$, there is a fairly narrow range for the optimum sensor position that provides maximum inverse accuracy; this was completely confirmed by the inverse treatment. As the number of sensors is increased successively by one, initially ($N = 2$) there is a sharp increase in the optimum value for the planning criterion and there are substantial changes in the optimum sensor coordinates. The region for locating the sensors expands considerably. Further increase in the number of sensors gives little increase in the criterion, so any further increase in that number is undesirable from the accuracy viewpoint. This is also confirmed by the inverse treatment (Fig. 2).

We also examined how the planning results were affected by the number of spline-approximation parameters and by conductivity variations.

Figure 3 shows how the planning criterion varies with the coordinate for one sensor with $m = 4$-7; as the number of parameters increases, ψ_{max} increases by several orders of magnitude, while the optimum sensor coordinate varies only slightly for $m = 5$-7.

These results confirm that optimum planning is viable with this criterion; if the exact values are specified as a priori information on the unknown conductivity, the locally optimal plan becomes exact, as is evident from the comparison of the plan and the inverse treatment.

Figure 4 shows the planning criterion as a function of sensor coordinate for various conductivities; the initial $\lambda(T)$ was perturbed via $\tilde{\lambda}(T) = a \times \lambda(T)$, and it was found that altering

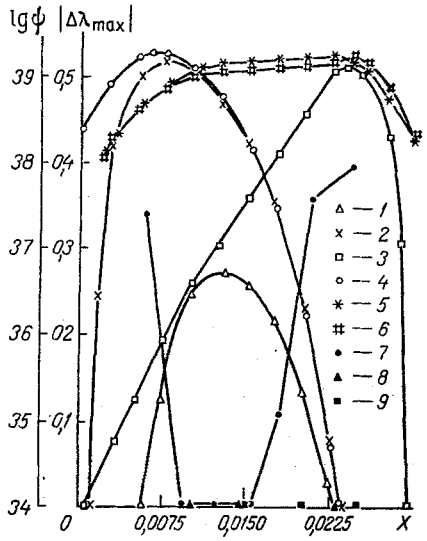

Fig. 2. Dependence of the planning criterion and maximal relative error in recovering the conductivity on the sensor position; for ψ: 1) N = 1; 2) N = 2, $0 < X_1 < 0.03$, $X_2 = X_2^*$; 3) N = 2, $X_1 = X_1^*$, $0 < X_2 < 0.03$; 4) N = 3, $0 < X_1 < 0.03$, $X_2 = X_2^*$, $X_3 = X_3^*$; 5) N = 3, $X_1 = X_1^*$, $0 < X_2 < 0.03$, $X_3 = X_3^*$; 6) N = 3, $X_1 = X_1^*$, $X_2 = X_2^*$, $0 < X_3 < 0.03$; for $|\Delta\lambda_{max}|$: 7) N = 1; 8) N = 2, $X_1 = X_1^*$, $0 < X_2 < 0.03$; 9) N = 3, $X_1 = X_1^*$, $0 < X_2 < 0.03$; $X_3 = X_3^*$.

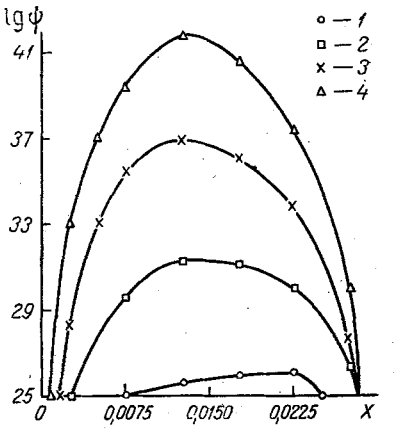

Fig. 3. Dependence of planning criterion on sensor position for various numbers of parameters: 1) m = 4; 2) 5; 3) 6; 4) 7.

$\lambda(T)$ by a factor three altered ψ by several orders of magnitude, while the optimum coordinate was only slightly altered. This shows that locally optimal planning can be used with little a priori information on the unknown characteristics.

The planning results agree well with the parametric analysis of the inverse accuracy also under other conditions of heat transfer at the boundaries and initial approximations for the functions.

A similar analysis can be based on the errors in the measurements if one simulates the error laws, which is the subject of a separate research.

NOTATION

T, temperature; x, coordinate; τ, time; τ_m, duration; b, plate thickness; $T_0(x)$, initial temperature distribution; C(T), bulk specific heat; $\lambda(T)$, thermal conductivity; $g_1(\tau)$, $g_2(\tau)$, boundary functions; I, functional; N, number of sensors; $f_i(\tau)$, $i = \overline{1, N}$, temperature measurements; $\bar{X} = \{X_i\}$, $i = \overline{1, N}$, sensor coordinate vector; X_i^*, $i = \overline{1, N}$, optimal sensor coordinates;

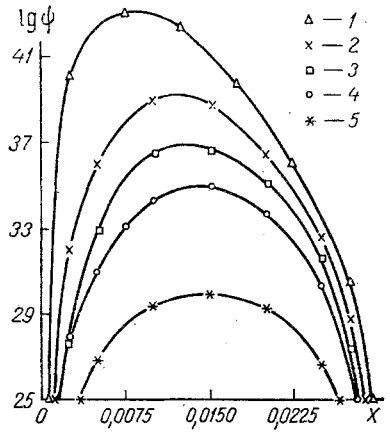

Fig. 4. Dependence of planning criterion on sensor position for various modes of variation in the a priori behavior of the conductivity $\lambda(T) = a \times \lambda(T)$: 1) $a = 1/3$; 2) 2/3; 3) 1; 4) 3/2; 5) 3.

ξ, measurement plan; λ_k, $k = \overline{1, m}$, approximation coefficients; φ_k, $k = \overline{1, m}$, basis functions; ν, conjugate variable; θ, temperature increment; $F(\xi)$, normalized Fisher information matrix; ϑ, Fisher matrix element; $\theta_k(x,\tau)$, $k = \overline{1, m}$, sensitivity functions; $\mu_i(\tau)$, $i = \overline{1, N}$, measurement performance functions; ψ, optimality criterion; $|\Delta\lambda_{max}|$, maximum relative error in recovering conductivity. Superscripts and subscripts: l, iteration number; i, measurement point number; j and k, parameter numbers.

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